Lecture 5

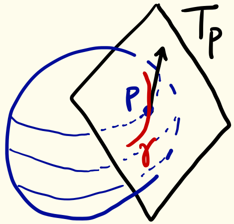
Geodesics and covariant derivatives

• Poll?

Recap

Where were we?

Tangent space



T_p = space of directional derivatives at p.

Coordinate basis of T_p

$$\boxed{\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\,\partial_\mu}$$

Vector transformation law

$$\boxed{V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu}\,V^\nu}$$

Cotangent space $T_p^* = \{$ space of linear maps $\omega : T_p \longrightarrow \mathbb{R}\}$

One-form transformation law

$$\boxed{\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\,\omega_\mu}$$

A (k,ℓ) **Tensor** is a linear map $T: \underbrace{T_p \otimes \cdots \otimes T_p}_{k} \otimes \underbrace{T_p^* \otimes \cdots \otimes T_p^*}_{\ell} \longrightarrow \mathbb{R}$

basis of T_p

$$T = \underbrace{T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}}_{\text{components in coordinate basis}} \; \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_\ell}$$

basis of T_p^*

Change of coordinates

$$\boxed{T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_\ell'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \frac{\partial x^{\nu_\ell}}{\partial x^{\nu_\ell'}} \; T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}}$$

The metric is a $(0,2)$ symmetric tensor that determines the line element:

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu$$

metric components

- inverse metric: $g^{\alpha\beta}$, $g^{\alpha\beta} g_{\beta\mu} = g_{\mu\beta} g^{\beta\alpha} = \delta^\alpha_\mu$
- determinant: $g = \det g_{\alpha\beta} \neq 0 \iff$ non-degenerate metric

Normal coordinates (at p)

$\exists$ coordinates $x^{\hat{\mu}}$ such that

$$\boxed{\begin{aligned} g_{\hat{\mu}\hat{\nu}}(p) &= \eta_{\hat{\mu}\hat{\nu}} \\ \partial_{\hat{\alpha}} g_{\hat{\mu}\hat{\nu}}(p) &= 0 \end{aligned}}$$

Differential Forms

A p-form is a $(0,p)$ tensor that is completely anti-symmetric.

- **Wedge product**

$$\boxed{(A\wedge B)_{\mu_1\cdots\mu_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1\cdots\mu_p} B_{\mu_{p+1}\cdots\mu_{p+q}]}}$$

- **Exterior derivative**

$$\boxed{(dA)_{\mu_1\cdots\mu_{p+1}} = (p+1)\,\partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]}} \quad \Rightarrow \quad \boxed{d^2 = 0}$$

- **Integration**

n-form

$$\int_{U\subset M} \alpha \equiv \int_{\phi(U)\subset \mathbb{R}^n} h\, dx^1 \cdots dx^n \qquad \alpha = h\, dx^1\wedge dx^2\wedge \cdots \wedge dx^n$$

n-dimensional

coordinate chart x^μ

The volume form or Levi-Civita tensor

If the manifold is equipped with a metric, then there is a special n-form ϵ such that

signature $\begin{cases} s=0 & \text{Euc.} \\ s=1 & \text{Lor} \end{cases}$

$$\boxed{\epsilon_{\mu_1 \cdots \mu_n} \epsilon_{\nu_1 \cdots \nu_n} g^{\mu_1 \nu_1} \cdots g^{\mu_n \nu_n} = (-1)^s \, n!}$$

$$\Rightarrow \quad \epsilon = \sqrt{|g|} \, \frac{1}{n!} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \equiv \sqrt{|g|} \, d^n x$$

Levi-Civita symbol $= \begin{cases} +1 & \text{if } (\mu_1 \cdots \mu_n) \text{ is an even perm. of } (0 1 \ldots n-1) \\ -1 & \text{'' '' '' odd '' ''} \\ 0 & \text{otherwise} \end{cases}$

We can then define integrals over M:

$$\int_M \epsilon \, \psi(x) = \int_M \psi \sqrt{|g|} \, d^n x$$

function (0-form)

Hodge duality

$$\underbrace{(\ast A)}_{(n-p)\text{-form}}{}_{\mu_1 \cdots \mu_{n-p}} \equiv \frac{1}{p!} \underbrace{\epsilon}_{\text{Volume form}}{}^{\nu_1 \cdots \nu_p}{}_{\mu_1 \cdots \mu_{n-p}} \overbrace{A}^{p\text{-form}}{}_{\nu_1 \cdots \nu_p}$$

Exercise : Show that

1) $\ast \ast A = A \, (-1)^{s + p(n-p)}$

2) Maxwell equations :

$$\boxed{\begin{aligned} dF &= 0 \\ d(\ast F) &= \ast J \end{aligned}}$$

metric only enters via $\ast$

$A = A_\mu \, dx^\mu$ 1-form

$F = dA = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu$

$J = J_\mu \, dx^\mu$

Generalized Stokes theorem

p-form

$$\int_{\Sigma} d\alpha = \int_{\partial\Sigma} \alpha$$

(p+1)-dimensional manifold

boundary of Σ

Exercise:

- $p = 0$ $\qquad \displaystyle\int_a^b \partial_x f(x)\,dx = f(b) - f(a)$
- $p = 1$ and surface $\Sigma \subset \mathbb{R}^3$ $\rightarrow$ Stokes theorem
- $\Sigma \subset \mathbb{R}^n$ region and $p = n-1$

  Hint: $\alpha = * \beta$

  (n-1) form $\qquad$ 1-form

Covariant Derivatives

How to generalize partial derivatives in a coordinate invariant way ?

$$\partial_\mu V^\nu \longrightarrow \nabla_\mu V^\nu$$

We require :

① Covariance : ∇ maps (k, l) tensor to $(k, l+1)$ tensors

② Linearity : $\nabla(T+S) = \nabla T + \nabla S$

③ Leibniz rule : $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$

④ Commute with contractions : $\nabla_\mu (T^\lambda{}_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho} \iff \nabla(\delta^\mu{}_\nu) = 0$

⑤ Reduces to partial der. on scalars : $\nabla_\mu \phi = \partial_\mu \phi$

Covariant derivative of a vector

② + ③ suggest

$$\boxed{\underbrace{\nabla_\mu V^\nu}_{(1,1)\text{ tensor}} = \underbrace{\partial_\mu V^\nu}_{\substack{\text{not}\\ \text{a tensor}}} + \Gamma^\nu_{\mu\lambda} V^\lambda}$$

not a tensor

Connection coefficients

Exercise: show that under coordinate transformations:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}$$

Covariant derivative of a 1-form

⑤ $$\nabla_\mu (\omega_\alpha V^\alpha) = \partial_\mu (\omega_\alpha V^\alpha)$$

③ $$(\nabla_\mu \omega_\alpha) V^\alpha + \omega_\alpha \underbrace{\nabla_\mu V^\alpha}_{\cancel{\partial_\mu V^\alpha} + \Gamma^\alpha_{\mu\nu} V^\nu} = (\partial_\mu \omega_\alpha) V^\alpha + \cancel{\omega_\alpha \partial_\mu V^\alpha}$$

$$\Rightarrow \cancel{V^\alpha} \left(\nabla_\mu \omega_\alpha - \partial_\mu \omega_\alpha + \Gamma^\lambda_{\mu\alpha} \omega_\lambda \right) = 0 \qquad \forall V^\alpha$$

$$\Rightarrow \boxed{\nabla_\mu \omega_\alpha = \partial_\mu \omega_\alpha - \Gamma^\lambda_{\mu\alpha} \omega_\lambda}$$

Covariant derivative of a tensor

$$\nabla_{\sigma} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = \partial_{\sigma} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$$

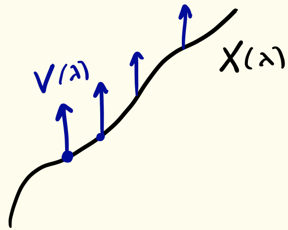
$$+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1 \lambda \mu_3 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \cdots$$

$$- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1 \cdots \mu_k}{}_{\lambda \nu_2 \cdots \nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \lambda \nu_3 \cdots \nu_l} - \cdots$$

$$\equiv T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l ; \sigma}$$

Parallel transport

How to transport a vector along a curve?



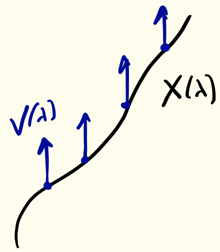
Impose that the cov. der. along the curve vanishes:

$$\boxed{\frac{D}{d\lambda} V^\nu \equiv \frac{dx^\mu}{d\lambda} \nabla_\mu V^\nu = 0}$$

Similarly for tensors : e.g.: $\frac{D}{d\lambda} T^{\mu\nu}{}_{\lambda} = 0$

Parallel transport

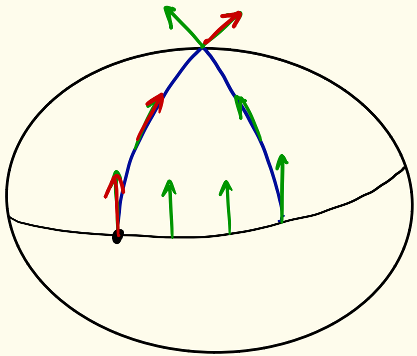
How to transport a vector along a curve ?



Impose that the covariant derivative along the curve vanishes :

$$\boxed{\frac{D}{d\lambda} V^\nu \equiv \frac{dx^\mu}{d\lambda} \nabla_\mu V^\nu = 0}$$

Similarly for tensors.



In the presence of curvature, parallel transport depends on the path.

Torsion is a $(1,2)$ tensor.

We can think of it as a map from $T_p \otimes T_p \longrightarrow T_p$:

$$\boxed{T(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y]}$$

where $\nabla_X \equiv X^\mu \nabla_\mu$

In components,

$$T^\alpha{}_{\mu\nu} X^\mu Y^\nu = X^\mu \nabla_\mu Y^\alpha - Y^\mu \nabla_\mu X^\alpha - [X,Y]^\alpha$$

$$= \left(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} \right) X^\mu Y^\nu$$

$$\boxed{T^\alpha{}_{\mu\nu} = 2 \, \Gamma^\alpha_{[\mu\nu]}}$$

The Levi-Civita connection

There is a unique torsion free and metric compatible connection.

$$T^{\lambda}_{\mu\nu}=0 \iff \Gamma^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{(\mu\nu)} \qquad\qquad \nabla_\rho g_{\mu\nu}=0$$

$$
\begin{aligned}
+\quad & \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \cancel{\Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu}} - \cancel{\Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda}} = 0 \\
-\quad & \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} - \cancel{\Gamma^{\lambda}_{\mu\rho} g_{\nu\lambda}} = 0 \\
-\quad & \nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \cancel{\Gamma^{\lambda}_{\nu\rho} g_{\lambda\mu}} - \Gamma^{\lambda}_{\nu\mu} g_{\rho\lambda} = 0
\end{aligned}
$$

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} = 0$$

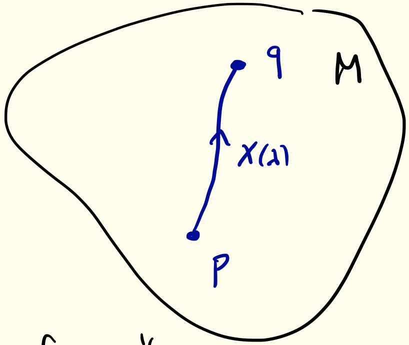
$g^{\sigma\rho}$

$$\boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)}$$

Christoffel symbols

Geodesics

A timelike geodesic maximizes the proper time between 2 points:



$$\tau = \int d\lambda \sqrt{- \underbrace{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}_{f}}$$

Trick: Notice that $\tau = \int d\lambda \sqrt{-f} \Rightarrow \delta\tau = -\frac{1}{2}\int (-f)^{-1/2}\, \delta f \, d\lambda$

For $\lambda =$ proper time $\Rightarrow f = -1$ and $\delta\tau = -\frac{1}{2}\int \delta f \, d\tau$

Therefore $\delta\tau = 0 \iff \boxed{\delta I = 0}$

with $\boxed{I \equiv \frac{1}{2} \int g_{\mu\nu} \, \dot{x}^\mu \, \dot{x}^\nu \, d\tau}$, $\dot{} \equiv \frac{d}{d\tau}$

$$\boxed{I \equiv \frac{1}{2}\int g_{\mu\nu}(x)\,\dot{x}^{\mu}\,\dot{x}^{\nu}\,d\tau} \quad , \quad \bullet \equiv \frac{d}{d\tau}$$

integrate by parts

$$\delta I = \frac{1}{2}\int\Big(\partial_\sigma g_{\mu\nu}\,\dot{x}^{\mu}\dot{x}^{\nu}\,\delta x^{\sigma} + g_{\mu\nu}\,\delta\dot{x}^{\mu}\,\dot{x}^{\nu} + \underbrace{-(g_{\mu\nu}\,\dot{x}^{\mu})\,\delta\dot{x}^{\nu}}_{-\delta x^{\nu}\left(g_{\mu\nu}\,\ddot{x}^{\mu} + \partial_\sigma g_{\mu\nu}\,\dot{x}^{\sigma}\,\dot{x}^{\mu}\right)}\Big)\,d\tau$$

$$= \cdots =$$

$$= -\int\left(\ddot{x}^{\sigma} + \Gamma^{\sigma}_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu}\right)\delta x_{\sigma}\,d\tau$$

christoffel symbols

$$\Rightarrow \boxed{\ddot{x}^{\sigma} + \Gamma^{\sigma}_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu} = 0} \iff \frac{D}{d\lambda}\dot{x}^{\sigma} = \dot{x}^{\mu}\nabla_{\mu}\dot{x}^{\sigma} = 0$$

geodesic equation

The tangent vector is parallel transported.

Charged test particle

vector potential

Action : $S = -m \int d\tau + q \int A_\mu dx^\mu$

metric enters here

Exercise : derive the Euler-Lagrange equations :

$$\boxed{m\left(\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\,\dot{x}^\rho \dot{x}^\sigma\right) = q\, F^\mu{}_\nu\, \dot{x}^\nu}$$

$\bullet \equiv \dfrac{d}{d\tau}$

proper time

Wisdom of the day

We are what we do repeatedly.

Good habits

Health
- Daily Physical exercise
- Sleeping well
- Eating "

- Making your bed
- Write a plan for the day

Bad habits

- Youtube / Tiktok before bedtime.

Lecture 6

The Riemann curvature tensor

- Poll

Recap

Where were we?

Covariant Derivatives

How to generalize partial derivatives in a coordinate invariant way?

$$\partial_\mu V^\nu \longrightarrow \nabla_\mu V^\nu$$

We require:

1. Covariance : ∇ maps (k,ℓ) tensor to $(k,\ell+1)$ tensors
2. Linearity : $\nabla(T+S) = \nabla T + \nabla S$
3. Leibniz rule : $\nabla(T\otimes S) = (\nabla T)\otimes S + T\otimes(\nabla S)$
4. Commute with contractions : $\nabla_\mu (T^\lambda{}_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho} \iff \nabla(\delta^\mu{}_\nu) = 0$
5. Reduces to partial der. on scalars : $\nabla_\mu \phi = \partial_\mu \phi$

Covariant derivative of a vector

not a tensor

$$\boxed{\underbrace{\nabla_\mu V^\nu}_{(1,1)\text{ tensor}} = \underbrace{\partial_\mu V^\nu}_{\text{not a tensor}} + \Gamma^\nu_{\mu\lambda} V^\lambda}$$

Connection coefficients

Covariant derivative of a tensor

$$\nabla_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} = \partial_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

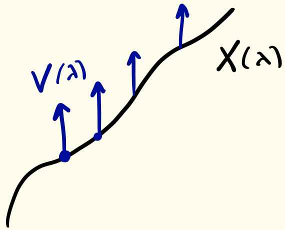
$$+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1 \lambda \mu_3 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} + \cdots$$

$$- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1 \cdots \mu_k}{}_{\lambda \nu_2 \cdots \nu_\ell} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \lambda \nu_3 \cdots \nu_\ell} - \cdots$$

Parallel transport

How to transport a vector along a curve ?

Impose that the cov. der. along the curve vanishes:



$$\boxed{\frac{D}{d\lambda} V^\nu \equiv \frac{dx^\mu}{d\lambda} \nabla_\mu V^\nu = 0}$$

Similarly for tensors :e.g.: $\frac{D}{d\lambda} T^{\mu\nu}{}_\lambda = 0$

The Levi-Civita connection

There is a unique torsion free and metric compatible connection.

$$T^{\lambda}_{\mu\nu} = 2\,\Gamma^{\lambda}_{[\mu\nu]} = 0 \qquad\qquad \nabla_\rho\, g_{\mu\nu} = 0$$

$$\Updownarrow$$

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{(\mu\nu)}$$

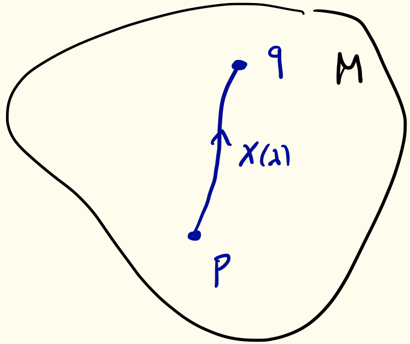
$$\Rightarrow \quad \boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\, g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)}$$

Christoffel symbols

Geodesics

A timelike geodesic maximizes the proper time between 2 points:

$$\tau = \int d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$



$\bullet \equiv \frac{d}{d\tau} \leftarrow$ proper time

Christoffel symbols

$$\delta\tau = 0 \Rightarrow \boxed{\ddot{x}^\sigma + \Gamma^\sigma_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0} \Leftrightarrow \frac{D}{d\tau}\dot{x}^\sigma = \dot{x}^\mu\nabla_\mu\dot{x}^\sigma = 0$$

geodesic equation

The tangent vector is parallel transported.

$$\boxed{\ddot{x}^{\sigma} + \Gamma^{\sigma}_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu} = 0}$$

$$\bullet \equiv \frac{d}{d\tau}$$

Generic parameter : $x^{\mu}(\alpha(\tau))$ $\quad \frac{d}{d\tau} = \dot{\alpha}\,\frac{d}{d\alpha}$

Exercise : show that

$$\frac{d^2x^{\sigma}}{d\alpha^2} + \Gamma^{\sigma}_{\mu\nu}\,\frac{dx^{\mu}}{d\alpha}\,\frac{dx^{\nu}}{d\alpha} = -\frac{\ddot{\alpha}}{\dot{\alpha}^2}\,\frac{dx^{\sigma}}{d\alpha}$$

$\ddot{\alpha} = 0 \iff \alpha = a\tau + b \iff \alpha$ is an affine parameter.

Massless particles move along null geodesics.

proper time vanishes for null geodesics.

It is convenient to use a parameter λ such that:

$$p^\mu = \frac{dx^\mu}{d\lambda} \qquad \text{4-momentum}$$

[massive particle

$$m\frac{dx^\mu}{d\tau} = p^\mu$$]

Geodesic equation:

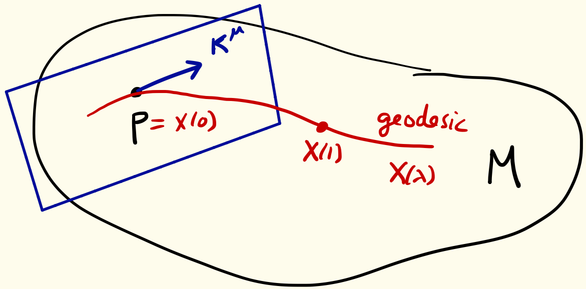
$$\boxed{p^\mu \nabla_\mu p^\nu = 0}$$

Local energy:

$$E = -p_\mu U^\mu_{obs}$$

The exponential map is a map from the tangent space T_p to the manifold.

$$\left.\frac{dx^\mu}{d\lambda}\right|_{\lambda=0} = k^\mu \in T_p$$



$$\exp_p(k) = x(\lambda=1)$$

This map is one-to-one in a neighbourhood of the origin of $T_p = \mathbb{R}^n$.

Riemann normal coordinates

1. Choose a basis $\{e_{\hat{\mu}}\}$ of T_p such that $g(e_{\hat{\mu}}, e_{\hat{\nu}}) = \eta_{\hat{\mu}\hat{\nu}}$.

Minkowski metric

$$k = k^{\hat{\mu}} e_{\hat{\mu}} \in T_p$$

2. Use the exponential map to define a coordinate system in the neighbourhood of p.

$$q = \exp_p(k) \quad \Rightarrow \quad \text{coordinates } x^{\hat{\mu}} \text{ of } q = k^{\hat{\mu}}$$

In this coordinate system, geodesics are given by $x^{\hat{\mu}}(\lambda) = \lambda k^{\hat{\mu}}$

$$\Rightarrow \quad \frac{d^2 x^{\hat{\mu}}}{d\lambda^2} = 0 \quad \Rightarrow \quad \Gamma^{\hat{\mu}}_{\hat{\sigma}\hat{\rho}} = 0 \quad \Rightarrow \quad \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} = \nabla_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} = 0$$

Plan for today

- The Riemann curvature tensor
- Ricci tensor, Ricci scalar, Weyl tensor, Einstein tensor
- Symmetries and Killing vectors

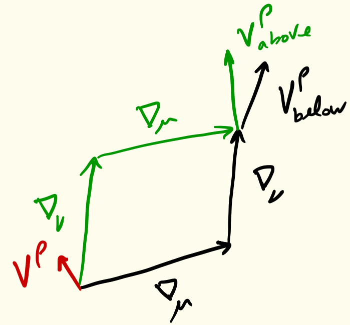
The Riemann Curvature Tensor

In the presence of curvature covariant derivatives do not commute:

$$\boxed{[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma}$$

Riemann tensor

(assuming torsion $= 0$)



Exercise: show that

$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}}$$

From now on, we will consider the Christoffel connection

$$\boxed{\Gamma_{\mu\nu}^{\sigma} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)}$$

$\exists$ coordinate system x^μ such that $g_{\mu\nu}(x) = \eta_{\mu\nu}$ $\iff$ $R^\rho{}_{\sigma\mu\nu} = 0$

Properties of the Riemann tensor

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu}$$

In normal coordinates at a point p

$$\left[g_{\hat\mu\hat\nu}(p) = \eta_{\hat\mu\hat\nu} \,, \quad \Gamma^{\hat\rho}_{\hat\mu\hat\nu}(p) = 0 \atop \partial_{\hat\rho} g_{\hat\mu\hat\nu}(p) = 0 \right]$$

$$R_{\hat\rho\hat\sigma\hat\mu\hat\nu} = g_{\hat\rho\hat\lambda}\left(\partial_{\hat\mu} \Gamma^{\hat\lambda}_{\hat\nu\hat\sigma} - \partial_{\hat\nu} \Gamma^{\hat\lambda}_{\hat\mu\hat\sigma} \right) = \ldots =$$

$$= \frac{1}{2}\left(\partial_{\hat\mu}\partial_{\hat\sigma} g_{\hat\rho\hat\nu} - \partial_{\hat\mu}\partial_{\hat\rho} g_{\hat\nu\hat\sigma} - \partial_{\hat\nu}\partial_{\hat\sigma} g_{\hat\rho\hat\mu} + \partial_{\hat\nu}\partial_{\hat\rho} g_{\hat\mu\hat\sigma} \right)$$

$$\boxed{R_{\rho\sigma\mu\nu} = - R_{\rho\sigma\nu\mu}}$$

$$\boxed{R_{\rho\sigma\mu\nu} = - R_{\sigma\rho\mu\nu}}$$

tensor eqs valid in any coord. system

$$\boxed{R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}}$$

$$\boxed{R_{\rho[\sigma\mu\nu]} = 0}$$

Properties of the Riemann tensor

In normal coordinates at a point p

$$\left[g_{\hat\mu\hat\nu}(p) = \eta_{\hat\mu\hat\nu} \,, \quad \partial_{\hat\rho} g_{\hat\mu\hat\nu}(p) = 0 \quad \Gamma^{\hat\rho}_{\hat\mu\hat\nu}(p) = 0 \right]$$

$$\nabla_{\hat\lambda} R_{\hat\rho\hat\sigma\hat\mu\hat\nu} = \partial_{\hat\lambda} R_{\hat\rho\hat\sigma\hat\mu\hat\nu}$$

$$= \partial_{\hat\lambda}\left[g_{\hat\rho\hat\theta}\left(\partial_{\hat\mu} \underbrace{\Gamma^{\hat\theta}_{\hat\nu\hat\sigma}}_{\frac{1}{2} g^{\hat\theta\hat\alpha}\left(\partial_{\hat\nu} g_{\hat\sigma\hat\alpha} + \partial_{\hat\sigma} g_{\hat\nu\hat\alpha} - \partial_{\hat\alpha} g_{\hat\nu\hat\sigma}\right)} - \partial_{\hat\nu}\Gamma^{\hat\theta}_{\hat\mu\hat\sigma} + \cancelto{0}{\Gamma\Gamma - \Gamma\Gamma} \right)\right]$$

$$= \frac{1}{2}\partial_{\hat\lambda}\left(\partial_{\hat\mu}\partial_{\hat\sigma} g_{\hat\rho\hat\nu} - \partial_{\hat\mu}\partial_{\hat\rho} g_{\hat\nu\hat\sigma} - \partial_{\hat\nu}\partial_{\hat\sigma} g_{\hat\rho\hat\mu} + \partial_{\hat\nu}\partial_{\hat\rho} g_{\hat\mu\hat\sigma} \right)$$

$$\Rightarrow \quad \boxed{\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0} \qquad \text{Bianchi identity}$$

Ricci tensor :

$$\boxed{R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}}$$

$$R_{\mu\nu} = R_{\nu\mu}$$

Ricci scalar :

$$\boxed{R \equiv g^{\mu\nu} R_{\mu\nu}}$$

Weyl tensor :

[similar to : $\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{n} g_{\mu\nu} T^{\alpha}{}_{\alpha}$
so that $\tilde{T}_{\mu\nu} g^{\mu\nu} = 0$]

$$\boxed{C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right) + \frac{2}{(n-2)(n-1)} g_{\rho[\mu} g_{\nu]\sigma} R}$$

Same symmetries of Riemann tensor with all traces removed (check!)

Weyl transformation : $g_{\mu\nu}(x) \to \omega^2(x)\, g_{\mu\nu}(x)$

$$C^{\rho}{}_{\sigma\mu\nu}(x) \to C^{\rho}{}_{\sigma\mu\nu}(x) \qquad \text{invariant!}$$

<u>Einstein tensor</u> :

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}$$

$$g^{\rho\mu} g^{\sigma\nu} \quad \boxed{\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0} \qquad \text{Bianchi identity}$$

$$g^{\rho\mu} g^{\sigma\nu} \left(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} \right) = 0$$

$$\nabla_\lambda R - \nabla^\mu R_{\lambda\mu} - \nabla^\nu R_{\lambda\nu} = 0$$

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = 0$$

$$\Rightarrow \qquad \boxed{\nabla^\mu G_{\mu\nu} = 0}$$

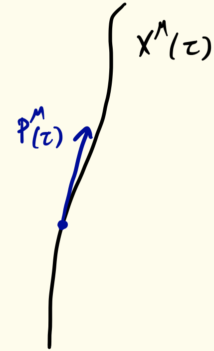
Symmetries and Killing vectors

Geodesic equation : $p^\mu \nabla_\mu p^\nu = 0$

$$\left[\begin{array}{lll} p^\mu = m \dfrac{dx^\mu}{d\tau} & & \text{massive} \\ p^\mu = \dfrac{dx^\mu}{d\lambda} & & \text{massless} \end{array}\right]$$

When do we have conserved quantities?

$$Q = K_\nu p^\nu$$



$$\frac{dQ}{d\tau} = 0 \iff p^\mu \nabla_\mu (K_\nu p^\nu) = 0$$

$$\iff p^\mu p^\nu \nabla_{(\mu} K_{\nu)} = 0$$

Killing vector field

Therefore, $\boxed{\nabla_{(\mu} K_{\nu)} = 0} \implies K_\mu p^\mu$ is conserved

Killing's equation

Wisdom of the day

The Stoic Challenge

See www.williambirvine.com for a modern stoic.